Return time entropies for a class of circle homeomorphisms

2004 J. Phys. A: Math. Gen. 37 6243

(http://iopscience.iop.org/0305-4470/37/24/003)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 6243–6250 PII: S0305-4470(04)69291-2

Return time entropies for a class of circle homeomorphisms

Nikola Buric and Kristina Todorović

Department of Physics and Mathematics, Faculty of Pharmacy, University of Beograd, Vojvode Stepe 450, 11000 Beograd, Yugoslavia

Received 19 September 2003
Published 2 June 2004
Online at stacks.iop.org/JPhysA/37/6243
doi:10.1088/0305-4470/37/24/003

Abstract
Poincaré recurrence for a class of circle maps is used to study the properties of the corresponding invariant measures. In the subcritical case, when the map is a diffeomorphism, the return time measure is smooth, and in the critical case, when the map is only a homeomorphism, the measure is only continuous. Furthermore, in the considered class of critical maps the behaviour of the return time entropy depends only on the tail in the continued fraction expansion of the rotation number.

PACS numbers: 05.45.−a, 45.05.+x

1. Introduction

Poincaré recurrence has recently gained renewed importance in the theory of dynamical systems. It is understood that the distribution of the return times could be efficiently used as a sensitive indicator of the statistical properties of the system dynamics (see, for example, [1–6] and the references therein).

The general mathematical framework of the problem is that of a measure space M with a reference measure μ and a transformation $T : M \to M$.

The first return time, in a measurable set $A \subset M$, of a point $x \in A$, is defined by

$$\tau_A(x) = \inf_k \{T^k(x) \in A\}. \tag{1}$$

The first return time τ_A for the set A, and the average return time $\langle \tau_A \rangle$ for the set A, are the following

$$\tau_A = \inf_{x \in A} \tau_A(x), \qquad \langle \tau_A \rangle = \int_A \tau_A(x)\, \mathrm{d}\mu_A(x) \tag{2}$$

where $\mu_A(x)$ is the conditional measure $\mu_A(B) = \mu(B)/\mu(A)$ for any $B \subseteq A$.

For ergodic systems the limit is the same for almost every point and in this case the average return time in the domain A is given by

$$\langle \tau_A \rangle = \frac{1}{\mu(A)} \tag{3}$$

where μ is the unique ergodic invariant measure. This is a consequence of the well-known Kac's lemma [7, 8] which can be formulated for general measure spaces as follows: [8] let T be an automorphism of a measure space (M, σ, μ) and $A \subseteq M$ with $\mu(A) > 0$. If T is mu-ergodic then

$$\int_A \tau_A(x)\, \mathrm{d}\mu_A = \frac{1}{\mu(A)}. \tag{4}$$

Motivated by the Kac lemma, our aim is to use the coarse-grained mean return times in order to study the corresponding return time entropies for a class of circle maps $T_{k,\Omega}$ of the following form:

$$x \rightarrow T_{k,\Omega}(x) = x + \Omega + \frac{k}{2\pi} f(2\pi x) \tag{5}$$

where $x \in \mathbf{R}/\mathbf{Z}$ and $k \in \mathbf{R}^+$ and $\Omega \in \mathbf{R}^+$ are the parameters of the map. The function $f(2\pi x)$ is a trigonometric polynomial such that the maps are monotonic for $k \in [0, 1)$ and non-invertible for $k > 1$. Furthermore, we shall restrict our attention to the class of maps where the coefficients in the trigonometric polynomial $f(2\pi x)$ are such that the map has an inflection point of the third degree at $x = 0$ for $k = 1$. Thus, all maps of the form (5) which we shall analyse cease to be diffeomorphisms via the same type of singularity. The details of the dynamics and the structure of the bifurcation diagram have been thoroughly studied for the simplest example of the form (5), i.e. the sine-circle map, given by $T_{k,\Omega}(x) = x + \Omega + \frac{k}{2\pi} \sin 2\pi x$. Other families of the form (5) are used to study the universality of the properties found for the sine-circle map.

2. Types of dynamics and the return times

Let us briefly recapitulate some of the properties of the circle maps (5). Here, we shall only deal with the subcritical and the critical cases.

For $k \leqslant 1$ the circle map $T_{k,\Omega}$ is an orientation preserving homeomorphism of the circle, and for $k < 1$ the map is a diffeomorphism. In any case, its topological properties are fixed by the rotation number ω defined by

$$\omega = \lim_{n\rightarrow\infty} \frac{\bar{T}^n(\theta)}{n} \tag{6}$$

where $\bar{T}$ is the lift of T on the real line, and $0 \leqslant \omega < 1$ for the definition to be unique.

The map $T_{k,\Omega}$, for $k < 1$ is conjugate to the linear rotation R_ω by the angle $\omega(k, \Omega)$, i.e. there is an invertible transformation of coordinate $\Theta = \Phi_{k,\Omega}(\theta)$ on the circle, such that $\Phi_{k,\Omega} \circ T_{k,\Omega} \circ \Phi_{k,\Omega}^{-1}\Theta = R_\omega\Theta \equiv \Theta + \omega$. The properties of the conjugation $\Phi_{k,\Omega}(\theta)$ depend on the arithmetic properties of ω. For a generic rotation number $\Phi_{k,\Omega}$ is a homeomorphism according to Denjoy's theorem [10]. Furthermore, if ω is a diophantine irrational the conjugation $\Phi_{k,\Omega}$ is a diffeomorphism [10].

A critical map $T_{k=1,\Omega}$ is only a homeomorphism of the circle but is still characterized by a unique rotation number.

The lines in the $(k < 1, \Omega)$ plane that correspond to maps with irrational rotation numbers become, for $k > 1$, domains such that a map in such a domain has at least one orbit with the corresponding irrational rotation number, but also has orbits with other rotation numbers belonging to an interval consistent with the P/Q tongues that overlap. Such a map also has many chaotic orbits with no rotation number [11–14]. The supercritical maps outside such domains have at least one and at most two attracting periodic orbits.

The numerically observed return times for $k \leqslant 1$ and diophantine rotation numbers ω are in agreement with a straightforward extension of Slater's theorem [9] for the three return times of the linear quasiperiodic rotation. In fact, for $k < 1$, it is enough to observe that any two points θ, $T^n_{k,\Omega}(\theta)$ in a connected interval A are mapped into $\Theta = \Phi_{k,\Omega}(\theta)$ and $R^n_\omega(\Theta)$ of the connected interval $\Phi_{k,\Omega}(A)$, where $\Phi_{k,\Omega}(\theta)$ is the linearizing transformation. The return times then follow from the Slater theorem, and the fact that any map of the form (5) with $k < 1$ is a diffeomorphism, so that, according to Denjoy's theorem, the linearizing transformation is an orientation preserving homeomorphism. The critical case $k = 1$ is more subtle since the critical map (5) is only a homeomorphism of the circle. However, numerical evidence, on irrational numbers with constant or just periodic tail of the continued fraction expansion, suggests that the statement about only three return times is true even in the critical case, with a possible reservation concerning the class of irrational numbers. Thus, we can conjecture that for any map in the class (5) with $k \leqslant 1$ and with sufficiently irrational rotation number, and for any connected interval, there are at most three different return times, one of them being the sum of the others.

Return times in the chaotic weakly supercritical standard circle map have been studied in [6]. Chaotic weakly supercritical maps lie in the region where many higher order Arnold tongues overlap. This is manifested by more then three return times into a sufficiently small interval. Numerical computations of the return times for the chaotic weakly supercritical maps in the class (5) lead to the same conclusions.

3. Return time entropy

The unique invariant ergodic measures, in the subcritical and in the critical cases, can be calculated using a sequence of partitions and the corresponding coarse-grained mean return times, as the partitions of the circle become finer. Guided by the Kac lemma, one is led to study, for an arbitrary map of the form (5), a positive mapping that associates with each measurable interval $A \in S^1$ the inverse of the mean return time for the interval A, i.e.

$$\mu_\tau(A) = \langle \tau_A \rangle^{-1}. \tag{7}$$

In the case of an ergodic system, according to the Kac lemma, this measure coincides with the unique ergodic invariant measure. Thus, the ergodic measure when the map is a homeomorphism can be approximated and studied by calculating the mean return times for a sequence of partitions. In the supercritical case there is no unique ergodic measure. However, the mean return times are still well defined, and it is interesting to investigate them for different sequences of partitions.

The density $\rho_{\mu_\tau}(x)$ of the mean return time is defined by

$$\mu_\tau(A) = \int_A \rho_{\mu_\tau}(x)\,\mathrm{d}x \tag{8}$$

where dx is the Lebesgue measure, (or more generally any reference measure).

The density $\rho_{\mu_\tau}(x)$ can be calculated with the help of nested intervals $\{A_i\}$ that shrink on the point x

$$\rho_{\mu_\tau}(x) = \lim_{A_i \to x} \left\langle \tau_{A_i} \right\rangle^{-1} = \lim_{A_i \to x} \mu_\tau(A_i). \tag{9}$$

Obviously, in the case the map is a homemorphism, the definition must not depend on the choice of the intervals. Thus, in order to calculate the density we can use the sequence of nested partitions that is best suited for the computation of the return times, like the partitions given by the continued fraction expansion, discussed in the next section.

Next we introduce the entropy of the return times and study it using the mean return times. Firstly, the return time entropy of a partition $\mathcal{P}_j$ is defined as

$$S_{\mu_\tau}(\mathcal{P}_j) = -\sum_i^{N_j} \mu_\tau(A_i)\ln\mu_\tau(A_i) = \sum_i^{N_j} \langle\tau_{A_i}\rangle^{-1} \ln\langle\tau_{A_i}\rangle \tag{10}$$

where N_j is the number of intervals in the jth partition, and $A_i, i = 1, 2, \ldots, N_j$ are the elements of $\mathcal{P}_j$. When $A_i \to 0$ together with $N_j \to \infty$ each of the summands in (10) goes to zero. Nevertheless, the sum could remain a finite number or even diverge to $+\infty$.

A partition $\mathcal{P}_k$ is said to be strictly greater than the partition $\mathcal{P}_j$ if it strictly refines the latter, i.e. for every $A_l \in \mathcal{P}_k$ there is an $A_m \in \mathcal{P}_j$ such that $A_l \subset A_m$. For notational convenience we will assume that the ordering between the partitions is mapped on the ordering of the partition indices. Suppose that $\mathcal{P} = \{\mathcal{P}_j\}$ is an ordered sequence of partitions. The return time entropy of the sequence is defined by the limit:

$$S_{\mu_\tau}(\mathcal{P}) = \lim_{j\to\infty} S_{\mu_\tau}(\mathcal{P}_j). \tag{11}$$

Note that we do not divide $S_{\mu_\tau}(\mathcal{P}_j)$ by N_j in the above limit. The reason for this will become clear later. Also, the dynamics of the map is reflected only via the properties of the return times. There is no special construction involved in the sequence of partitions $\mathcal{P}$. Any ordered sequence could appear in (11). The return time entropy of the map is now defined as the supremum of (11) over all ordered sequences of partitions:

$$S(\mu_\tau) = \sup_{\mathcal{P}}\{S_{\mu_\tau}(\mathcal{P})\}. \tag{12}$$

In order to classify the critical maps it will turn out to be useful to introduce a scaling coefficient α defined by the following condition:

$$\lim_{j\to\infty} \frac{S_{\mu_\tau}(\mathcal{P}_j)}{\alpha\ln N_j + \text{const}} = 1 \tag{13}$$

where const is an additive constant independent of j. If the limit (11) exists and is finite the scaling coefficient of the return time entropy is zero. If the return time entropy of the sequence of partitions diverges logarithmically the scaling coefficient is a finite positive number. If the divergence is faster than logarithmic the coefficient is infinite.

4. Continued fraction partition

In the case $k \leqslant 1$ when the map (5) is a homeomorphism, the invariant measure is unique, and, due to the Kac lemma, the limits in (11) do not depend on the sequence of partitions. On the other hand, the return times, and their relative weights, depend on the location and the size of the interval. However, there is a sequence of intervals, obtained by partitioning the circle with the iterates of an initial point θ_0, that is best suited for the analysis of the return times at a point θ_0 of the map with a given irrational rotation number ω. Furthermore, the union of the intervals formed by iterates of the point θ_0 gives a partition $\mathcal{P}_i$ of the circle, and the return times into various intervals of the partition $\mathcal{P}_i$ are the same.

To construct the partition, one considers the trajectory formed by the $q_{i-1} + q_i - 1$ successive iterates of a point θ_0, where p_i/q_i are rational continued fraction approximates of the ω. The q_ith and the q_{i-1}th iterate form the boundary of an interval $A_i(\theta_0)$ which

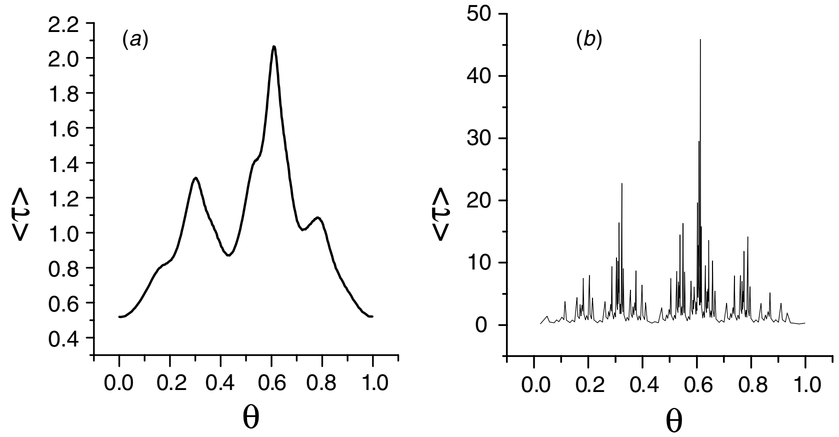

Figure 1. Illustration of the return time coarse-grained measure for the standard circle map, with a sufficiently fine partition, in (*a*) the subcritical ($k = 0.75$) and (*b*) the critical case ($k = 1$).

contains only the initial point θ_0 and no other points of the considered part of the trajectory. The points generated by $q_{i+1} + q_i - 1$ iterates will subdivide the intervals generated by $q_i + q_{i-1} - 1$ iterates. One obtains a sequence $A_{i+1} \subseteq A_i$ of intervals that converge to the point θ_0 on the orbit of $T_{k,\Omega}$. Calculating the return time τ_{A_i} for such a sequence of intervals $A_i, i \to \infty$ is best adopted to the calculation of the return time at the point $\theta_0 \in \bigcap_{i=1}^{\infty} A_i$ [6]. Suppose that the q_{i-1}th iterate is to the left of θ_0 which is to the left of the q_ith iterate. It is proved in [6] that: in the case of the standard homeomorphism of the circle, each of the points $\theta \in A_i(\theta_0)$ that are on the left side of θ_0 have a unique first return time equal to q_i, and points $\theta \in A_i(\theta_0)$ to the right of θ_0 also have a unique first return time equal to q_{i-1}. Thus, there are only two return times equal to q_{i-1} and q_i. This fact greatly reduces the number of iterations in the numerical computations of the mean return times into the intervals A_i. The return times into various intervals of a partition $\mathcal{P}_i$ are the same q_i and q_{i-1}. The proof relies on the orientation preservation of the homeomorphisms (5), and thus is valid for any map of this form with $k \leqslant 1$. Furthermore, it is obvious that for the homeomorphisms of the form (5) the return times for intervals in one partition do not depend on the initial point θ_0 that is used to construct the partition.

5. Numerical results

The main results of our computation are illustrated in figures 1, 2 and 3.

In figures 1(*a*) and (*b*) we plot, on the *y*-axis, the coarse-grained mean return time $\langle \tau_{A_i} \rangle$ (denoted by $\langle \tau \rangle$ in the figure) into the intervals of a partition $\mathcal{P}_j$. On the *x*-axis, the mid-points $\theta_i \in A_i$ of the intervals A_i are plotted. The partition is sufficiently fine, so that the figure illustrates the density of the unique invariant ergodic measure for the considered map. The main cost of computing the invariant measure via the return times is due to the computational time, significantly reduced by using the continued fraction partitions, complementary to the perturbative method where the main requirement is for a sufficient storage space.

For $k < 1$ the density of the return time measure is a smooth function (see figure 1(*a*)), and for the critical maps the density becomes singular (figure 1(*b*)). This is in agreement with the results in [15, 16]. Fractal properties of the ergodic measures for the critical circle

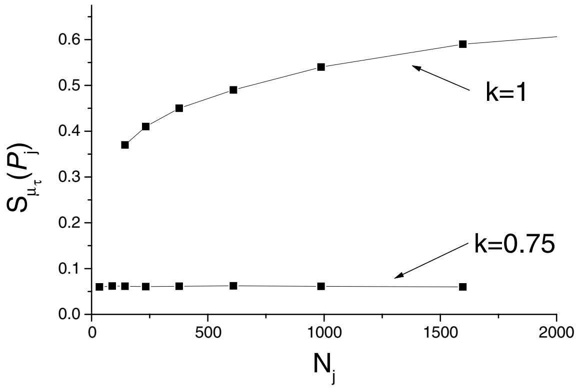

Figure 2. Illustration of the convergence of the entropy for the subcritical ($k = 0.75$) and the critical case. x-axes represent the number of elements in the partitions and y-axes represent the return time entropy of the partition. Both lines are for the standard circle map with the rotation number numerically equal to the inverse of the golden mean.

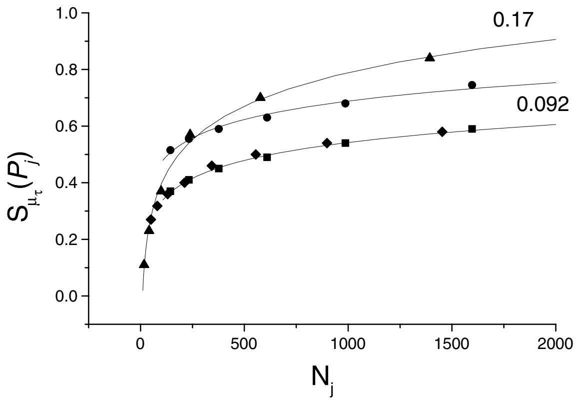

Figure 3. Illustration of the dependence of the return time entropy on the rotation number and the map. The axes are the same as in figure 2. The interpolated smooth curves represent $\alpha \ln N_j + \text{const}$, $\alpha = 0.092$ for boxes, diamonds and circles, and $\alpha = 0.17$ for triangles (see the main text).

maps (5) have been studied. There is strong evidence [16] that the class of critical maps with the same fractal spectrum as the invariant measure are characterized only by the rotation number (actually probably only by the tail in its continued fraction expansion [17]) and by the type of the singularity that induces the critical behaviour. Return times have been used by Afraimovich [18] to define a dimension in a construction similar to the construction of the Hausdorf dimension. However, this dimension is the same number for any circle map which is topologically conjugate to an irrational rotation.

The return time entropy is a well-defined quantity in the subcritical as well as the critical case. The entropy $S(\mu_\tau)$ does not depend on the sequences of partitions that are used for its numerical computation. The limit in its definition gives a finite number in the subcritical case. In the critical case the entropy of a sequence of partitions diverges logarithmically. Thus, the scaling coefficient α is zero in the subcritical case, and in the critical case it is a finite positive number, as is illustrated in figure 2, where $\alpha = 0.092$ for $k = 1$ and $\alpha = 0$ for $k = 0.75$, both for the standard circle map with the rotation number equal to the inverse of the golden mean. Finally, our main conclusion, illustrated in figure 3, is that in the critical case the scaling of the entropy α is independent of the map and the particular rotation number on an orbit of the uni-modular group. However, for the rotation numbers which are not related by a uni-modular transformation the scaling coefficient of the return time entropy is different. In figure 3, boxes correspond to the golden rotation number and diamonds to the rotation number $[0, 1, 2, 2, 1^{\infty}]$ and the standard circle map. Circles correspond to the golden rotation number but the map is given by $f(x) = \sin(2\pi x) + 0.1 \sin^3(2\pi x)$. In all three cases the coefficients α in $S_{\mu_\tau}(\mathcal{P}_j) = \alpha \ln N_j + \text{const}$ are the same, within the numerical error, and equal to $\alpha = 0.092$. On the other hand, triangles correspond to a rotation number $[0, 1, 2^{\infty}]$ and the standard circle map. For this tail, the coefficient of the logarithmic divergence is $\alpha = 0.17$ clearly different from $\alpha = 0.092$. This type of universality is also shared by the fractal spectra of the critical invariant curves.

6. Summary and conclusions

We have investigated the Poincaré recurrences for a class of homeomorphisms of the circle. The recurrences are used to study the return time entropies.

In the subcritical and the critical case only three return times are observed. This is theoretically and numerically justified in [6] for the case of the standard circle map. We found that the same property appears valid for any homeomorphism of the form (5). For diophantine rotation numbers ω this result can be presented as a corollary of Slater's theorem on the return times of linear irrational rotations.

The main technical tool that we have used is the special sequence of partitions, that gives a very simple proof of the observed return times, and enables efficient calculations of the return time measures and entropies.

An approximation to the invariant measure of the subcritical and critical maps and the diophantine rotation numbers is obtained from the average return times. Taking a sufficiently fine partition gives a simple method of computation of the invariant measure with an accuracy comparable to other methods. We have then defined and calculated the return time entropy. This quantity has the same universality properties as the fractal spectra of the critical invariant measure.

In the supercritical case there is no unique erqodic measure. However, the mean return times, and the coarse-grained return time entropies, are still well defined. It would be interesting to investigate them for different sequences of partitions, and different dynamical regimes of the supercritical maps. Another useful extension could be attempted in the analysis of the dynamics of area-preserving maps near critical invariant circles.

Acknowledgments

This work is an extension of the collaboration with Professor G Turchetti and Dr A Rampioni from the Department of Physics of the University of Bologna, to whom we are deeply indebted.

References

[1] Zaslavsky G M 2002 Dynamical traps *Physica* D **168–169** 292
[2] Chirikov B V and Shepelyansky D I 1999 Asymptotic statistics of Poincaré recurrences in Hamiltonian systems with divided phase space *Phys. Rev. Lett.* **82** 528

[3] Hirata M, Saussol B and Vaienti S 1999 Statistics of return times: a general framework and new applications *Commun. Math. Phys.* **206** 33
[4] Barreira L and Sausol B 2001 Hausdorf dimension of measures via Poincaré recurrence *Commun. Math. Phys.* **219** 443
[5] Burić N, Rampioni A, Turchetti G and Vaienti S Poincaré 2003 Recurrences for area-preserving maps *J. Phys. A: Math. Gen.* **36** L209
[6] Burić N, Rampioni A and Turchetti G 2003 Statistics of Poincaré recurrences for a class of smooth circle maps *Chaos, Soliton Fractals* at press
[7] Kac M 1947 On the notion of recurrence in discrete stochastic processes *Bull. Am. Math. Soc.* **53** 1002
[8] Corneld I P, Sinai Ya G and Fomin S V 1982 *Ergodic Theory* (New York: Springer)
[9] Slater N B 1967 Gaps and steps for the sequence $n\theta$ mod 1, *Proc. Camb. Phil. Soc.* **63** 1115
[10] Herman M 1979 Sur la conjugasion différentiable des difféomorphismes du cercle a des rotations *Publ. Math. IHES* **49** 5
[11] Boyland P L 1986 Bifurcations of circle maps: Arnold's tongues, bistability and rotation intervals *Commun. Math. Phys.* **106** 356
[12] Jensen M N, Bak P and Bohr T 1984 Transition to chaos by interaction of resonances in dissipative systems. I. Circle maps *Phys. Rev.* A **30** 1960
[13] MacKay R S and Tresser C 1986 Transition to topological chaos for circle maps *Physica* D **19** 206
[14] Rand D, Ostlung S, Sethna J and Siggia E D 1982 Universal transition from quasiperiodicity to chaos in dissipative systems *Phys. Rev. Lett.* **49** 132
[15] Khanin K M 1991 Universal estimates for critical circle maps *Chaos* **1** 181
[16] Hunt B R, Khanin K M, Sinai Ya G and Yorke J A 1996 Fractal properties of critical invariant curves *J. Stat. Phys.* **85** 261
[17] Burić N, Mudrinić M and Todorović 1998 Universal scaling of critical quasiperiodic orbits in a class of twist maps *J. Phys. A: Math. Gen.* **31** 7847
[18] Afraimovich V 1997 Pesin's dimension for Poincaré recurrences *Chaos* **7** 12